Quantum III, SCU, Fall 2023.

1. The γ Matrices:

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \qquad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

2. Generating Algebra: $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} I_4$, $\not{a}\not{b} + \not{b}\not{a} = 2a \cdot b$ with $\not{B} \equiv \gamma^\mu B_\mu$

3. Matrix Identities:

   (a) γ contraction:

   i. $\gamma^\mu \gamma_\mu = 4I_4$
   ii. $\gamma_\mu \gamma^\nu \gamma^\mu = -2\gamma^\nu$
   iii. $\gamma_\mu \gamma^\nu \gamma^\lambda \gamma^\mu = 4g^{\nu\lambda}$
   iv. $\gamma_\mu \gamma^\nu \gamma^\lambda \gamma^\sigma \gamma^\mu = -2\gamma^\sigma \gamma^\lambda \gamma^\nu$
   v. $\gamma_\mu \not{a} \gamma^\mu = -2\not{a}$
   vi. $\gamma_\mu \not{a}\not{b} \gamma^\mu = 4a \cdot b$
   vii. $\gamma_\mu \not{a}\not{b}\not{c} \gamma^\mu = -2\not{c}\not{b}\not{a}$

   (b) Trace Technology:

   i. $\mathrm{tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$
   ii. $\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4\left(g^{\rho\sigma}g^{\mu\nu} - g^{\nu\sigma}g^{\mu\rho} + g^{\mu\sigma}g^{\nu\rho}\right)$
   iii. $\mathrm{tr}(\gamma^5) = 0$
   iv. $\mathrm{tr}\left(\not{a}\not{b}\right) = 4(a \cdot b)$
   v. $\mathrm{tr}\left(\not{a}\not{b}\not{c}\not{d}\right) = 4\left[(a\cdot b)(c\cdot d) - (a\cdot c)(b\cdot d) + (a\cdot d)(b\cdot c)\right]$
   vi. $\mathrm{tr}\left(\gamma^5\not{a}\not{b}\right) = 0$
   vii. $\mathrm{tr}\left(\gamma^5\not{a}\not{b}\not{c}\not{d}\right) = 4i\epsilon^{\mu\nu\rho\sigma}a_\mu b_\nu c_\rho d_\sigma$
   viii. $\mathrm{tr}(\gamma^5\gamma^\mu) = 0$
   ix. The trace of the product of an odd number of γ's is zero.

Summary Table :

	electrons e^-	positrons e^+	Photons γ
Represented wave functions	$\psi(x) = N e^{-i p_\mu x^\mu/\hbar} u^s(p)$ $p_\mu = (E/c, \vec{p})$	$\psi(x) = N e^{i p_\mu x^\mu/\hbar} v^s(p)$ $p_\mu = (E/c, \vec{p})$	$A^\mu(x) = N e^{-i p_\mu x^\mu/\hbar} \epsilon^\mu_s(p)$
Satisfy	$(\gamma^\mu p_\mu - mc)u(p) = 0$ (momentum-space)	$(\gamma^\mu p_\mu + mc)v(p) = 0$	$\epsilon^\mu p_\mu = 0$
Adjoint:	$\bar{u}(\gamma^\mu p_\mu - mc) = 0$ $\bar{u} = u^\dagger \gamma^0$	$\bar{v}(\gamma^\mu p_\mu + mc) = 0$ $\bar{v} = v^\dagger \gamma^0$	– –
Orthogonality	$\bar{u}^{(1)} u^{(2)} = 0$	$\bar{v}^{(1)} v^{(2)} = 0$	$\epsilon^\mu_{(1)} \epsilon^*_{\mu(2)} = 0$
Normalized	$u^\dagger u = 2\lvert E\rvert/c$ $\bar{u} u = 2mc$	$v^\dagger v = 2\lvert E\rvert/c$ $\bar{v} v = -2mc$	$\epsilon^{\mu *} \epsilon_\mu = 1$
Completeness	$\sum_{s=1,2} u^{(s)} \bar{u}^{(s)} = (\gamma^\mu p_\mu + mc)$	$\sum_{s=1,2} v^{(s)} \bar{v}^{(s)} = (\gamma^\mu p_\mu - mc)$	$\sum \epsilon_{(s)i} \epsilon^*_{(s)j} = \delta_{ij} - \hat{p}_i \hat{p}_j$
Spinors	$u^{(1)} = N \begin{pmatrix} 1 \\ 0 \\ cp_z/(E+mc^2) \\ c(p_x+ip_y)/(E+mc^2) \end{pmatrix}$ $u^{(2)} = N \begin{pmatrix} 0 \\ 1 \\ \frac{c(p_x-ip_y)}{E+mc^2} \\ -cp_z/(E+mc^2) \end{pmatrix}$	$v^{(1)} = N \begin{pmatrix} \frac{c(p_x-ip_y)}{E+mc^2} \\ \frac{-cp_z}{E+mc^2} \\ 0 \\ 1 \end{pmatrix}$ $v^{(2)} = -N \begin{pmatrix} \frac{cp_z}{E+mc^2} \\ \frac{c(p_x+ip_y)}{E+mc^2} \\ -1 \\ 0 \end{pmatrix}$	choose : $\vec{p} = (0, 0, \lvert\vec{p}\rvert)\,\hat{z}$ $\epsilon^1 = (1,0,0)$ $\epsilon^2 = (0,1,0)$ (four components, but only 3 needed) Full Form Polarization Vector $\epsilon^\mu_\pm = \frac{1}{\sqrt{2}}(0, 1, \pm i, 0)$