Quantum III Equation Sheet — Fall 2023 — Ivar Rydstrom

Probability Current

$$\vec{J}(x,t) = \frac{i\hbar}{2m}\left(\psi\frac{\partial\psi^*}{\partial x} - \psi^*\frac{\partial\psi}{\partial x}\right)$$

$$\frac{\partial P(t)}{\partial t} = -\frac{\partial J(x,t)}{\partial x}$$ ← Continuity Equation.

Translation Operator

$$T(d\vec{x}) = 1 - i\vec{K}\cdot d\vec{x}'$$

$$T(d\vec{x})|x'\rangle = |x' + d\vec{x}'\rangle$$

Parity Operator

$$\hat{\Pi}|x\rangle = |x'\rangle = |-x\rangle$$

Unitary $U^{-1} = U^\dagger$

Hermitian $H = H^\dagger$

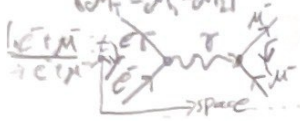
$[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$ "Commutator" "Lie"

$\{\hat{A},\hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A}$ "Anti-Commutator" "Lie Algebra"

PHOTON-MEDIATED

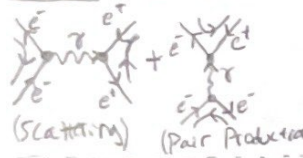
$(\mathcal{M}_T = \mathcal{M}_1 - \mathcal{M}_2)$

1. Møller: $e^- + \mu^- \to e^- + \mu^-$



2. Møller: $e^- + e^- \to e^- + e^-$

3. Bhabha: $e^- + e^+ \to e^- + e^-$

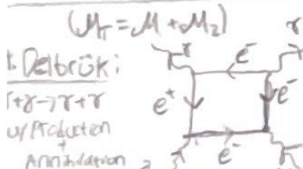


(Scattering) (Pair Production)

ELECTRON-MEDIATED

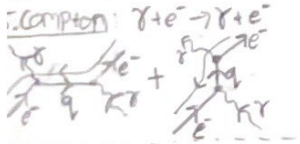
$(\mathcal{M}_T = \mathcal{M} + \mathcal{M}_2)$

1. Delbrück: $\gamma + \gamma \to \gamma + \gamma$ w/ Production + Annihilation

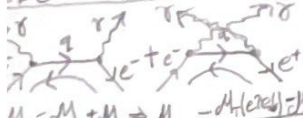


(might be wrong across)

2. Compton: $\gamma + e^- \to \gamma + e^-$



3. Electron + Positron Pair Annihilation



$\mathcal{M}_T = \mathcal{M}_1 + \mathcal{M}_2 \Rightarrow \mathcal{M}_{\text{singlet}} = \frac{\mathcal{M}_1(\uparrow\downarrow) - \mathcal{M}_1(\downarrow\uparrow)}{\sqrt{2}}$

$\mathcal{M}_{\text{Final}} = \frac{\mathcal{M}_{\text{singlet}}(\uparrow\downarrow) - \mathcal{M}_{\text{singlet}}(\downarrow\uparrow)}{\sqrt{2}}$

→ "Classical" Quantum Scattering

$d\Omega = \sin\theta\, d\theta\, d\phi$ ← Solid Angle

$|\mathcal{M}|^2 = |f(\theta)|^2 = D(\theta) = \frac{d\sigma}{d\Omega}$ — "differential cross-section"

3D SE: $-\frac{\hbar^2}{2m}\frac{d^2u(r)}{dr^2} + \left[V(r) + \underbrace{\frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}}_{\text{"centrifugal term"}}\right]u(r) = Eu(r)$

$u(r) = rR(r)$

Fourier's Trick → $\int_0^\pi P_l(\cos\theta)P_{l'}(\cos\theta)\sin\theta\, d\theta = \left(\frac{2}{2l+1}\right)\delta_{l,l'}$

Relativistic Quantum Scattering

Signature: $(1,-1,-1,-1)$

Relativity: $d\tau = \frac{1}{\gamma}dt$

$\eta^\mu = \left(\gamma c, \frac{d\vec{x}}{d\tau}\right) = \gamma\left(c, \frac{d\vec{x}}{dt}\right)$

$p^\mu = \left(\frac{E}{c}, \vec{p}\right) \to \hat{p}^\mu = i\hbar\partial^\mu$

$J^\mu = (c\rho, \vec{J})$ $X^\mu = (ct, \vec{x})$

$\gamma^0 = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$ $\vec{\gamma} = \begin{pmatrix}0&\vec{\sigma}\\-\vec{\sigma}&0\end{pmatrix}$

$\gamma^\mu = (\beta, \beta\vec{\alpha})$, $\beta = \begin{pmatrix}I&0\\0&-I\end{pmatrix}$, $\vec{\alpha} = \begin{pmatrix}0&\vec{\sigma}\\\vec{\sigma}&0\end{pmatrix}$

Generating Algebra: $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}I_4$

$\gamma_L = \frac{1}{\sqrt{1-v^2/c^2}}$ $\beta_L = \frac{v}{c}$

$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\nabla\right)$

$\partial^\mu = \frac{\partial}{\partial x_\mu} = \left(-\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right)$

$\Box = \partial_\mu\partial^\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$

Klein-Gordon Equation

$$\left[\Box + \left(\frac{mc}{\hbar}\right)^2\right]\psi(t,x) = 0$$

(Spin-0 Bosons, ±Energy, disaster)

Dirac Equation

$$i\hbar\gamma^\mu\partial_\mu\psi - mc\psi = 0$$

$(\gamma^\mu p_\mu - mc)u(P) = 0$

← Derivative-free Dirac.

$\bar{\psi} \equiv \psi^\dagger\gamma^0$ $\not{\partial} \equiv \gamma^\mu\partial_\mu$

Timelike: $a_\mu a^\mu > 0$

Spacelike: $a_\mu a^\mu < 0$

Light like: $a_\mu a^\mu = 0$

$g_{\mu\nu} = \begin{bmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{bmatrix}$ $x_\mu = g_{\mu\nu}x^\nu$, $x_\nu g^{\mu\nu} = x^\mu$

$\sigma_x = \begin{pmatrix}0&1\\1&0\end{pmatrix}$, $\sigma_y = \begin{pmatrix}0&-i\\i&0\end{pmatrix}$, $\sigma_z = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$

Levi-Civita symbol

$\epsilon_{ijk} = \begin{cases}1: \text{cyclic } (i,j,k)\\-1: \text{anti-cyclic } (k,j,i)\\0: \text{any repeating}\end{cases}$

$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$

$\vec{A}\cdot\vec{B} = \sum_i(\vec{A}\cdot\vec{B})_i = \sum_i A_iB_i$

$\vec{A}\times\vec{B} = \sum_i(\vec{A}\times\vec{B})_i = \sum_{j,k=1}^{3}\epsilon_{ijk}A_jB_k$

$A_\mu B^\mu = A\cdot B$

Dirac Gamma Matrices:

$\gamma^0 = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}$ $\gamma^1 = \begin{pmatrix}0&0&0&1\\0&0&1&0\\0&-1&0&0\\-1&0&0&0\end{pmatrix}$

$\gamma^2 = \begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&i&0&0\\-i&0&0&0\end{pmatrix}$ $\gamma^3 = \begin{pmatrix}0&0&1&0\\0&0&0&-1\\-1&0&0&0\\0&1&0&0\end{pmatrix}$

$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}$

$(\gamma^\mu)^2 = I_4$, $\{\gamma^5, \gamma^\mu\} = 0$ $\gamma^\mu\gamma_\mu = 4I_4$

$\gamma^\mu\gamma^\nu\gamma_\mu = -2\gamma^\nu$ $\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4g^{\nu\rho}I_4$

$\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu = -2\gamma^\sigma\gamma^\rho\gamma^\nu$

JONES VECTOR

$\epsilon_1 = \frac{1}{\sqrt{2}}(1,1,0)$ $\epsilon_1\cdot\epsilon_1 + \epsilon_2\cdot\epsilon_2 = -2(\epsilon_1\cdot\epsilon_2)$

$\epsilon_2 = \frac{1}{\sqrt{2}}(1,-i,0)$ $\epsilon_1^*\epsilon_2 - \epsilon_1\epsilon_2^* = 2i(\epsilon_1\times\epsilon_2)$

$d\Gamma = |\mathcal{M}|^2\frac{S}{2E_1m_1}\left[\frac{Cd^3\vec{P}_2}{(2\pi)^3 2E_2}\right]\left[\frac{Cd^3\vec{P}_3}{(2\pi)^3 2E_3}\right]\cdots\left[\frac{Cd^3\vec{P}_n}{(2\pi)^3 2E_n}\right](2\pi)^4\delta^4(P_1 - P_2 - \dots - P_n)$

Fermi's Golden Rule (Decay rate). $P_i = \left(\frac{E_i}{c}, \vec{P}\right)$, $S = \frac{1}{J_1!}\frac{1}{J_2!}\cdots$ where J_i = # of particle i in decay

$\tau = \frac{1}{\Gamma}$

$\delta(k\vec{x}) = \frac{1}{|k|}\delta(x)$

$$\sigma = \int|\mathcal{M}|^2\frac{\hbar^2 S}{4\sqrt{(P_1\cdot P_2 - m_1^2m_2^2c^4)}}\left[\frac{Cd^3\vec{P}_3}{(2\pi)^3 2E_3}\right]\cdots\left[\frac{Cd^3\vec{P}_n}{(2\pi)^3 2E_n}\right](2\pi)^4\delta^4(\underbrace{P_1 + P_2}_{\text{in}} - \underbrace{P_3 - \dots - P_n}_{\text{out}})$$

Fermi's Golden Rule (Scattering σ) $\Rightarrow 1 + 2 \to 3 + 4 + \dots + n$.

$d^3\vec{P}_3 = |\vec{P}_3|^2 dP_3 d\Omega \Rightarrow$ yields $\left(\frac{d\sigma}{d\Omega} = D(\theta)\right)$

$\sum_{\text{spins}}[\bar{u}(a)\Lambda_1 u(b)][\bar{u}(a)\Lambda_2 u(b)]^* = \text{Tr}(\Lambda_1(\not{P}_b + m_bc)\bar{\Lambda}_1(\not{P}_a + m_ac))$ ← Cosmin's Trick (normalized state)

"Quantum Hardware" AKA Formalism + Notation

$\sum_\alpha|\alpha\rangle\langle\alpha| = 1$ ← "Closure" identity

Integral form: $\int_{\text{all space}}|x'\rangle\langle x'|d^3x' = 1$ ←

$\langle\beta|\hat{x}|\alpha\rangle = \langle\alpha|\hat{x}^\dagger|\beta\rangle^*$

Legendre Polynomial Orthogonality

$\int_{-1}^{1}P_l(x)P_{l'}(x)\,dx = \frac{2}{2l+1}\delta_{l,l'}$

$\int_0^\pi P_l(\cos\theta)P_{l'}(\cos\theta)\sin\theta\, d\theta = \frac{2}{2l+1}\delta_{l,l'}$

$\langle P'|x'\rangle = \langle x'|P'\rangle^* = \frac{1}{\sqrt{2\pi\hbar}}e^{\frac{-iP'x'}{\hbar}}$

Feynman Diagrams

Photon Propagator: $\frac{-ig_{\mu\nu}}{q^2}$ e^+e^- Propagator: $\frac{i(\gamma^\mu q_\mu + mc)}{q^2 - m^2c^2}$

	Toward S	Away
Particle	$u(P)$ ↗	$\bar{u}(P)$ ↗
Anti-Particle	$\bar{v}(P)$ ↙	$v(P)$ ↙
γ	ϵ^μ	$\epsilon^{*\mu}$ (Polarization vector)

Vertex coupling constant $\left(ie\sqrt{\frac{4\pi}{\hbar c}}\gamma^\mu\right)$ (for every vertex)

Conservation Laws: $(2\pi)^4\delta^4(P_1 + P_2 + \dots)$ (one for each vertex)

Integration: $\int\frac{d^4q}{(2\pi)^4}$ (one for each internal momentum (mediator))

in = +, out = −, anti-particle opposite. S = source, M = mediator, V = vertex ($\not{P}$, $\not{q}$) ...

$i(2\pi)^4\delta^4(P_{in} - P_{out})\mathcal{M} = \int[\bar{S}(V\bar{S})](M)[S(V\bar{S})]\epsilon_\mu^*\epsilon_\nu^*\frac{d^4q}{(2\pi)^4}$

(Always Alternate Index up / Index Down) (Repeated indices) (Polarization vectors for γs)

Spinors:

$u(1) = N\begin{pmatrix}1\\0\\\frac{cP_z}{E+mc^2}\\\frac{c(P_x+iP_y)}{E+mc^2}\end{pmatrix}$ SU → $u(2) = N\begin{pmatrix}0\\1\\\frac{c(P_x-iP_y)}{E+mc^2}\\-\frac{cP_z}{E+mc^2}\end{pmatrix}$ SD →

$v(2) = N\begin{pmatrix}\frac{cP_z}{E+mc^2}\\\frac{c(P_x+iP_y)}{E+mc^2}\\1\\0\end{pmatrix}$ SD → $v(1) = N\begin{pmatrix}\frac{c(P_x-iP_y)}{E+mc^2}\\-\frac{cP_z}{E+mc^2}\\0\\1\end{pmatrix}$ SU →

$N = \sqrt{\frac{|E| + mc^2}{c}}$

Dirac Spinor Transformations:

$\psi \to \psi' = S\psi$, $S_{4\times4} = a_+ + a_-\gamma^0\gamma^1$

$a_\pm = \pm\sqrt{\frac{1}{2}(\gamma \pm 1)}\, I_{4\times4}$ ← Lorentz γ. $[\gamma^5, S] = 0$

"A is invariant" ⟺