Physics 121 (Quantum I) Equation Sheet

Winter 2023

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Statistics

$\langle f(j)\rangle = \sum_{j=0}^{\infty} f(j)\cdot P(j)$ ← expectation value of func. $f(j)$ | $\langle x\rangle = \int_{-\infty}^{\infty} x\,\rho(x)\,dx$ | $\langle f(x)\rangle = \int_{-\infty}^{\infty} f(x)\rho(x)\,dx$

$\langle Q\rangle = \int \Psi^* \hat{Q}\,\Psi\,dx = \langle \Psi|\hat{Q}\Psi\rangle$

$\sigma^2 = \langle(\Delta j)^2\rangle$ ← variance

$\sigma = \sqrt{\langle j^2\rangle - \langle j\rangle^2}$ ← standard deviation

$P_{ab} = \int_a^b \rho(x)\,dx$ ← Prob. of x in a→b ($P_{-\infty\,\infty} = 1$)

prob. density

$\int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx = 1 = \langle\Psi|\Psi\rangle$ ← Normalization

Probability density

$[x, L_z^2] = -i\hbar y$ (permutated)

$\hat{P} = -i\hbar\frac{\partial}{\partial x}$ $\quad \hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)$

$\langle \hat{P}\rangle = m\frac{d\langle x\rangle}{dt}$

$\frac{d}{dt}\langle\hat{Q}\rangle = \frac{i}{\hbar}\langle[\hat{H},\hat{Q}]\rangle + \langle\frac{\partial\hat{Q}}{\partial t}\rangle$

$\hat{H} = \frac{\hat{p}^2}{2m} + V$ $\quad \boxed{\hat{H}\Psi = E\Psi}$

$[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$ ← commutator

$\sigma_A^2\sigma_B^2 \geq \left(\frac{1}{2i}\langle[\hat{A},\hat{B}]\rangle\right)^2$ $\quad \sigma_x\sigma_p \geq \frac{\hbar}{2}$ $\quad [\hat{x},\hat{p}] = i\hbar$

$[A+B] = [A,B] + (B)$ $\quad [AB,C] = A[B,C] + [A,C]B$

Infinite Square Well

$V(x) = \begin{cases} 0 & 0<x<a \\ \infty & \text{else} \end{cases}$ (for Schrödinger's)

$\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)$, $\quad E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$

$\Psi(x,t) = \sum_{n=1}^{\infty} c_n\psi_n(x)e^{-\frac{iE_n}{\hbar}t}$ $\quad c_n = \int \psi_n^*(x)f(x)\,dx$

(Any TISE solution) $\quad c_n = \langle f_n|\Psi\rangle$

→ If explicitly finding $\langle x\rangle, \langle p\rangle$ → Plug in $\hat{a}_\pm$ form not x

Quantum Harmonic Oscillator

$(F = -kx)$ $\quad V = \frac{1}{2}kx^2$ $\quad \omega = \sqrt{\frac{k}{m}}$ $\quad \langle T\rangle = \frac{1}{2}m\langle v\rangle^2$

$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$ $\quad E_n = (n+\frac{1}{2})\hbar\omega$ $\quad \hat{H} = \hbar\omega(\hat{a}_\pm\hat{a}_\mp \pm \frac{1}{2})$

$\psi_n(x) = \frac{1}{\sqrt{n!}}(\hat{a}_+)^n\psi_0(x)$ $\quad \hat{a}_+\psi_n = \sqrt{n+1}\,\psi_{n+1}$ $\quad [\hat{a}_-,\hat{a}_+] = 1$

$\hat{a}_\pm = \frac{1}{\sqrt{2\hbar m\omega}}(\mp i\hat{p} + m\omega x)$ $\quad \hat{a}_-\psi_n = \sqrt{n}\,\psi_{n-1}$

$x = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_+ + \hat{a}_-)$ $\quad \hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a}_+ - \hat{a}_-)$

Hydrogen: $n = 1, 2, \ldots$; $l = 0, 1, 2, \ldots, (n-1)$; $m = -l, \ldots, l$

$\lambda = \frac{h}{P}$ $\quad P = \hbar k$

Quantum Free Particle

$V=0$, $\Psi(x,t) = Ae^{ik(x-\frac{\hbar k}{2m}t)} + Be^{-ik(x+\frac{\hbar k}{2m}t)}$ (R) (L)

$\Psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(k)\,e^{i(kx - \frac{\hbar k^2}{2m}t)}dk$ $\quad k \equiv \pm\frac{\sqrt{2mE}}{\hbar}$

$\phi(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Psi(x,0)e^{-ikx}dx$

Fourier Transform $x \to k$

$T = \frac{|\text{coeff out}|^2}{|\text{coeff in}|^2}$ $\quad R = \frac{|\text{coeff back}|^2}{|\text{coeff in}|^2}$ $\quad R+T = 1$

Bound State: $E < V(x)$ as $x \to \pm\infty$ $(E<0)$

Scattering State: $E > V(x)$ as $x \to \pm\infty$ $(E>0)$

Plancherel's Theorem

$\Phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} e^{-\frac{ipx}{\hbar}}\Psi(x,t)\,dx$

$\Psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} e^{\frac{ipx}{\hbar}}\Phi(p,t)\,dp$

3D Quantum: $\Psi_{nlm}(r,\theta,\phi) = R_{nl}(r)Y_l^m(\theta,\phi)$

$Y_l^m(\theta,\phi) = \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}\,e^{im\phi}P_l^m(\cos\theta)$

$P_l^m(x) = (-1)^m(1-x^2)^{m/2}\left(\frac{d}{dx}\right)^m P_l(x)$

$P_l(x) = \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l(x^2-1)^l$

Hydrogen: $E_n = \frac{E_1}{n^2}$, $n = 1, 2, 3\ldots$

$R_{nl}(r) = \frac{1}{r}\rho^{l+1}e^{-\rho}V(\rho)$ $\quad \rho = \frac{1}{a_0 n}$

$V(\rho) = \sum c_j x^j$, $\quad c_{j+1} = \frac{2(j+l+1-n)}{(j+1)(j+2l+2)}c_j$

$\Psi_{nlm} = \sqrt{\left(\frac{2}{na}\right)^3\frac{(n-l-1)!}{2n(n+l)!}}\,e^{-\frac{r}{na}}\left(\frac{2r}{na}\right)^l\left[L_{n-l-1}^{2l+1}\left(\frac{2r}{na}\right)\right]Y_l^m(\theta,\phi)$ — $R(r)$

$L_q(x) = \frac{e^x}{q!}\left(\frac{d}{dx}\right)^q(e^{-x}x^q)$

Non-Degenerate Perturbation

$E_n^1 = \langle\psi_n^0|H'|\psi_n^0\rangle$

$\psi_n^1 = \sum_{m\neq n}\frac{\langle\psi_m^0|H'|\psi_n^0\rangle}{E_n^0 - E_m^0}\psi_m^0$

$E_n^2 = \sum_{m\neq n}\frac{|\langle\psi_m^0|H'|\psi_n^0\rangle|^2}{E_n^0 - E_m^0}$

$E_\pm = \frac{1}{2}\left[W_{aa} + W_{bb} \pm \sqrt{(W_{aa}-W_{bb})^2 + 4|W_{ab}|^2}\right]$

$W_{ij} = \langle\psi_i^0|H'|\psi_j^0\rangle \to (i,j = a,b)$

Schrödinger's Equation

$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi$

TISE: $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi$

$\phi(t) = e^{-\frac{iE}{\hbar}t}$

Stationary state

$|\Psi(x,t)|^2$ = time independent $= |\psi|^2$

not $\psi(x,t)$ itself.

Hermite Polynomials

$H_n(\xi) = (-1)^n e^{\xi^2}\left(\frac{d}{d\xi}\right)^n e^{-\xi^2}$

Linear Algebra

$A^{-1} = \frac{1}{|A|}\tilde{C}_{ij}$ $\quad A^\dagger = \tilde{A}^*$

Unitary: $U^{-1} = U^\dagger$

Hermitian: $H = H^\dagger$ $\quad \langle f|\hat{Q}f\rangle = \langle\hat{Q}f|f\rangle$

Orthogonal: $\tilde{O} = O^{-1}$

Angular Momentum / Spin

$m = -l, \ldots, l$ $\quad l \Rightarrow s$ $\quad$ also possible L_x, L_y values

$\hat{L}^2|lm\rangle = \hbar^2 l(l+1)|lm\rangle$

$\hat{L}_z|lm\rangle = \hbar m|lm\rangle \Rightarrow \langle\hat{L}_z\rangle = \hbar m$

$[\hat{L}_x,\hat{L}_y] = i\hbar\hat{L}_z$ permutate... $[\hat{L}_y,\hat{L}_z]$, $[\hat{L}_z,\hat{L}_x]$

$\hat{L}_\pm|lm\rangle = \hbar\sqrt{l(l+1) - m(m\pm1)}\,|l\,m\pm1\rangle$

$\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$

$[\hat{L}^2,\hat{L}_{x,y,z}] = 0$ ← L^2 commutes w/ L_x, L_y, L_z

$\hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y$ $\quad \hat{L}_x = \frac{1}{2}(\hat{L}_+ + \hat{L}_-)$ $\quad \hat{L}_y = \frac{1}{2i}(\hat{L}_+ - \hat{L}_-)$

$\langle L^2\rangle = \hbar^2 l(l+1)$ $\quad L_z = -i\hbar\frac{\partial}{\partial\phi}$

Spin

$|\uparrow\rangle = |\frac{1}{2}\,\frac{1}{2}\rangle = \chi_+$ $\quad |\downarrow\rangle = |\frac{1}{2}\,{-\frac{1}{2}}\rangle = \chi_-$

$\hat{S}_z = \frac{\hbar}{2}\sigma_z = \frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ $\quad \hat{S}^2 = \frac{3}{4}\hbar^2\begin{pmatrix}1&0\\0&1\end{pmatrix}$

$\hat{S}_x = \frac{\hbar}{2}\sigma_x = \frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}$ $\quad \hat{S}_+ = \hbar\begin{pmatrix}0&1\\0&0\end{pmatrix}$

$\hat{S}_y = \frac{\hbar}{2}\sigma_y = \frac{\hbar}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix}$ $\quad \hat{S}_- = \hbar\begin{pmatrix}0&0\\1&0\end{pmatrix}$

WKB Approximation

$$P(x) \equiv \sqrt{2m(E - V(x))}$$

$$\int_0^a P(x)\,dx = n\pi\hbar \leftarrow \text{2 walls}$$

$$\int_0^{x_2} P(x)\,dx = \left(n - \tfrac{1}{4}\right)\pi\hbar \leftarrow \text{1 wall}$$

$$\int_{x_1}^{x_2} P(x)\,dx = \left(n - \tfrac{1}{2}\right)\pi\hbar \leftarrow \text{no walls.}$$

Gram-Schmidt Orthaginalization:

Given $\vec{A}, \vec{B}, \vec{C}$ orthog:

$$|\hat{e}_1\rangle = \frac{|A\rangle}{\|\vec{A}\|} \quad |e_2\rangle = |\vec{B}\rangle - \langle \hat{e}_1|B|\hat{e}_1\rangle, \ |\hat{e}_2\rangle = \frac{|e_2\rangle}{\|e_2\|}$$

$$|e_3\rangle = |\vec{C}\rangle - \langle \hat{e}_1|C\rangle|\hat{e}_1\rangle - \langle \hat{e}_2|C|\hat{e}_2\rangle \quad |\hat{e}_3\rangle = \frac{|e_3\rangle}{\|e_3\|}$$

$$\int_0^\infty x^{2n} e^{-\frac{x^2}{a^2}}\,dx = \sqrt{\pi}\left(\frac{(2n)!}{n!}\right)\left(\frac{a}{2}\right)^{2n+1} \leftarrow \text{(gaussian)}$$

Taylor expansion:

$$f(a+h) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} h^n \quad h = (x-a)$$

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$$

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2}$$