Physics 104 Equation Sheet

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Taylor Expansion

$$f(a+h) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} h^n \leftarrow h=(x-a)$$

a-centered

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$\hookrightarrow e^{ix} = \cos x + i \sin x$

Euler's Equation

a. $f = f(y', x)$
a1. $f = f(y')$

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$$

b. $f = f(y, y')$

$$\frac{\partial f}{\partial x} - \frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right) = 0$$

Cylindrical $ds = \sqrt{dr^2 + r^2 d\theta^2 + dz^2}$

→ w/explicit constraints (Lagrange Multipliers)

$$\frac{\partial f}{\partial y_i} - \underbrace{\frac{d}{dx}\frac{\partial f}{\partial y_i'}}_{\text{(maybe 0)}} + \sum_i \lambda \frac{\partial g}{\partial y} = 0 \leftarrow g = 0 \leftarrow \text{constraint}$$

Lagrange Equation

$L = T - U$

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = 0$$

Hamiltonian

$$H = \sum \dot{q}_k p_k - L$$

$$p_k = \frac{\partial L}{\partial \dot{q}_k}$$

$$\frac{\partial H}{\partial p_k} = \dot{q}_k, \quad -\frac{\partial H}{\partial q_k} = \dot{p}_k$$

→ E/H conserved if no "t" in L $= \frac{\partial L}{\partial t} = 0$
"scleronomic"

Reference Frames

$$\underbrace{m\vec{a}_r}_{\text{rotating frame accel}} = \vec{F} - \underbrace{m\ddot{\vec{R}}_T}_{\text{translation accel.}} - \underbrace{m\dot{\vec{\omega}} \times \vec{r}}_{\text{angular } \alpha} - \underbrace{2m\vec{\omega} \times \vec{v}_r}_{\text{Coriolis}} - \underbrace{m\vec{\omega} \times (\vec{\omega} \times \vec{r})}_{\text{centrifugal}}$$

$\omega = (\omega\cos\lambda, 0, \omega\sin\lambda)$

Moment of Inertia / Rigid Bodies

$$I_{ij} = \sum_\alpha m\left(\delta_{ij} \sum_\alpha x_\alpha^2 - x_i x_j\right)$$

$$I_{ij} = \int \rho(\vec{r}) \left(\delta_{ij} \sum_\alpha x_\alpha^2 - x_i x_j\right) dV$$

$\sum_\alpha x_\alpha^2 = x^2 + y^2 + z^2$

$$\vec{r}_{cm} = \frac{\int \vec{r}\, dm}{M_{tot}}$$

$\{I\} \cdot \vec{\omega} = I \cdot \vec{\omega} \leftarrow$ eigenvalue prob.

Principal Moments of Inertia

$$\begin{pmatrix} I_{11} - I & I_{12} & \cdots \\ & I_{22} - I & \\ & & I_{33} - I \end{pmatrix} \cdot \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = 0$$

|| axis theorem $\rightarrow J_{ij} = I_{ij} + M \cdot (\delta_{ij} a^2 - a_i a_j)$

a = dist from CM to new axes

cm

$N = \frac{dL}{dt} \leftarrow L = I\omega$

ex: $a = (\_, \_, \_)$ r_{cm}?

Euler's Equations of Rotational Motion

$$I_1 \dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 = N_1$$
$$I_2 \dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 = N_2$$
$$I_3 \dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 = N_3$$

→ torques

for $I_1 = I_2 \neq I_3$
$\dot{\omega}_1 = +$
$\dot{\omega}_2 = -$
(=0 sum) + (=0 sum) = 0
$\eta = \omega_1 + i\omega_2$
$\dot{\eta} =$
$\eta(t) = Ae^{i\Omega t}$

$e^{i\theta} = \cos\theta + i\sin\theta$
ω_1 term, ω_2 term

Stability / Perturbation

$$\Omega_1 = \omega_1 \sqrt{\frac{(I_1 - I_3)(I_1 - I_2)}{I_2 I_3}}$$

$$\Omega_2 = \omega_2 \sqrt{\frac{(I_2 - I_1)(I_2 - I_3)}{I_1 I_3}}$$

$$\Omega_3 = \omega_3 \sqrt{\frac{(I_3 - I_2)(I_3 - I_1)}{I_1 I_2}}$$

Central Force Motion

$$\frac{\alpha}{r} = 1 + \varepsilon \cos\theta$$

$$\alpha = \frac{\ell^2}{\mu k}$$

$$\varepsilon = \sqrt{1 + \frac{2E\ell^2}{\mu k^2}}$$

$$r_{min} = a(1-\varepsilon) = \frac{\alpha}{1+\varepsilon}$$

$$\tau^2 = \frac{4\pi^2 \mu}{k} a^3$$

$$r_{max} = a(1+\varepsilon) = \frac{\alpha}{1-\varepsilon}$$

$$2a = r_{min} + r_{max}$$

$$E_{tot} = \frac{-k}{2a}$$

$$a = \frac{k}{2|E|} \quad b = \frac{\ell}{\sqrt{2\mu|E|}}$$

$$E_{tot} = T + U$$