Math 103

Advanced Linear Algebra
Midterm 1

Vector Space Axioms

1. $x+y=y+x$
2. $x+(y+z)=(x+y)+z$
3. $\vec{0}+\vec{x}=\vec{x}$, (there exists $\vec{0}$) ← $\exists\, \vec{0}\in V$
4. for every $\vec{x}$, there exists $(-\vec{x})$ s.t. $\vec{x}+(-\vec{x})=0$
5. $a(x+y)=ax+ay$
6. $(a+b)x=ax+bx$
7. $1\cdot\vec{x}=\vec{x}$ ← multiplicative identity
8. $ab(\vec{x})=a(b\vec{x})$

implicit: $\vec{x}\in V$, $\vec{y}\in V$, then $\vec{x}+\vec{y}\in V$ ← closure under addition
$c\vec{x}\in V$, then $\vec{x}\in V$ ← closure under scalar mult.

External Direct Sum

$V_1\oplus V_2=\{(\vec{v}_1,\vec{v}_2)\mid \vec{v}_1\in V_1,\ \vec{v}_2\in V_2\}$
Don't talk, separate

$\dim(V_1\oplus V_2)=\dim(V_1)+\dim(V_2)$

Internal Direct Sum

$U+W=\{\vec{u}+\vec{w}\mid \vec{u}\in U \,\&\, \vec{w}\in W\}$

kind of just $U+W$ is a basis for V, if U,W bases.
Talk, are together $\Rightarrow U,W\subseteq V$
$\Rightarrow U,W\in V$

↳ Every vector $\vec{v}$ can be written $\vec{v}=\vec{u}+\vec{w}$, if $V=U+W$

Check:
1. $\dim(U)+\dim(W)=\dim(V)$ ← spans; $\in U$, $\in W$
2. $\dim(U\cap W)=0$ ← unique. (LI)
$(U\cap W=\{\vec{0}\})$.

Backwards

Prove uniqueness!
$\vec{v}\in V$ & $\vec{v}'\in V$
$\Rightarrow \vec{v}=\vec{v}'$!!!

Subspace

1. $\vec{0}\in W$
2. Closure under addition
3. Closure under scalar multiplication.

Span + LI. = Basis

$\text{Span}\{\vec{v}_1\ldots\vec{v}_n\}=a_1\vec{v}_1+\ldots+a_n\vec{v}_n$.

LI: $a_1\vec{v}_1+\ldots+a_n\vec{v}_n=0\Rightarrow a_1\ldots a_n=0$

A Basis is a vectorspace which spans some other vectorspace and is LI.

Replacement Theorem:

of elements in a LI subset of V is less than or equal to # of elements in a span of V.

$\boxed{\text{LI}\leq\text{span}}$

Dimension = # of elements in a basis of V.

Goldilocks Theorems

Let $\dim(V)=n$

1. LI. subset of V w/ n elements = Basis
2. Spanning subset of V w/ n elements = Basis
3. a. Subspace (W) of V is finite dimensional.
   b. $\dim(W)\leq n$
   c. If $\dim(W)=n$, then $W=V$.

Linear Transformations

$T: V \to W$ (Implied: $T(\vec{0}) = \vec{0}$) ★ key!

Linear if: ① $T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2)$

② $T(c\vec{v}) = cT(\vec{v})$

Coordinate Vectors

$\{\vec{v}_1 \ldots \vec{v}_n\} = B$ = ordered basis for V.

$\vec{v} \in V$, $[\vec{v}]_B = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$

in B basis (could be in others..)

coordinate vector = linear combination of Basis vectors.

To form T:

1) $\boxed{\text{Apply T}}$ to each vector in basis of Domain

2) $\boxed{\text{Expand}}$ resulting vectors using basis of Codomain

3) $\boxed{\text{Collect}}$ coordinate vectors C for each vector. Those are columns of matrix $[T]_{CB}$.

Invertible Matrix Theorem

Assume $T: V \to W$ is linear.

$\text{Dim}(V) = \text{Dim}(W) = N$ (square matrix or not invertible)

One = All!

1. T is invertible
2. T is one-to-one
3. T is onto
4. If [T] is a matrix for T, then [T] is invertible
5. If [T] is a matrix for T, then columns of [T] are LI.
6. If [T] is matrix for T, then columns of [T] span $\mathbb{R}^n$

Bijective = one-to-one + onto

Rank + Kernel + Nullspace + Range, etc..

- Domain(T) = Basis for home T
- Range(T) = Basis for map $T: V \to W$.
- $\text{Rank}(T) = \dim(\text{range}(T))$ ← # of vectors in range (# of rows)
- Kernel = Nullspace (for Linear Transformations) (All vectors in domain which map to $\vec{0}$)

Rank-Nullity Theorem

$\dim(V) = \text{Rank}(T) + \dim(\text{kernel}(T))$

(Rank = dim(range))

Domain (start) = codomain (end) + zero (waste).

Basis of Range: Pivot columns of T

Basis of Null: reduce matrix. $= \vec{0}$

Multiply out as $x_1 + x_2 = 0$

$x_3 + x_4 = 0$

Then write in terms of non-pivot cols (free vars)

$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ \vdots \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ \vdots \end{bmatrix}$ ← coordinate vector is basis for Null.